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# Equivalent substitution in the control theory

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In this paper we study a problem of looking for an optimal solution of a system of the differential equations with a control and an optimized function. The system of differential equations is changed for two systems with the upper and lower approximation functions of a function on the right side of the initial differential system and the lower approximation of the optimized function in a region of attainability. The necessary conditions of optimality are sufficient for the substituted system.

A rule of an evaluation of an attainability set with the help of the positively definite functions is given in the second part of the paper.

**Key words.** Optimal control problems, optimal trajectories, convex functions, lower convex approximation, attainability set, convex analysis, linear function.

## 1 Introduction

Consider the following general problem of the control theory. Let us have a system of the differential equations

$$\dot{x}(t) = \varphi(x(t), u(t)), \quad x(0) = x_0, \quad (1)$$

and an optimized functional has a form

$$J(u) = \int_0^T f(x(\tau), u(\tau)) d\tau \rightarrow \inf_u, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t)$  takes values in  $U \subset \mathbb{R}^r$ , where  $U$  is a convex compact set in  $\mathbb{R}^r$ ,  $t \in [0, T]$ . We assume that the function  $f(\cdot, \cdot)$  is continuous and the function  $\varphi(\cdot, \cdot)$  is Lipschitz in all arguments, so that the system (1) satisfies the conditions of uniqueness for a solution passing through a point of the phase space. We consider the autonomous systems of the differential equations that does not restrict generality of consideration.

We have to find an optimal control function  $u(\cdot)$  that is a piecewise continuous differentiable vector-function from  $KC^1[0, T]$  with values in  $U$ , defined on the

segment  $[0, T]$ , for which the solution of the equation (1) gives the infimum of the functional  $J(\cdot)$ . We replace the optimization problem (2) on the following problem

$$J(u, t) = \int_0^t f(x(\tau), u(\tau)) d\tau \longrightarrow \inf_{u \in KC^1[0, T], t \in [0, T]}. \quad (3)$$

We assume that the derivatives  $u'(\cdot)$ , where they exist, are uniformly bounded in the norm, i.e.

$$\|u'(t)\| \leq C \quad \forall t \in \aleph_u[0, T],$$

where  $\aleph_u[0, T]$  is a set of the points on  $[0, T]$ , where derivatives of  $u(\cdot)$  exist. In this case the pointwise convergence of a sequence  $\{u_k(\cdot)\}$  on  $[0, T]$  is equivalent to the uniform convergence of the functions  $u_k(\cdot)$  on the set of continuity and, of course, is equivalent to the convergence in the metrics  $\rho$  of the space  $KC^1[0, T]$ , which is equal, by definition, to the metrics of the space  $C[0, T]$ , i.e.

$$\rho(u_1(t), u_2(t)) \stackrel{\text{def}}{=} \max_{t \in [0, T]} \|u_1(t) - u_2(t)\|.$$

Let us include into consideration all functions resulting from the pointwise limit. It is obvious that all the limit functions belong to a closed, bounded set of functions on  $[0, T]$ , which we denote by  $\mathcal{KC}^1[0, T]$ . The two functions from the set  $\mathcal{KC}^1[0, T]$  are equivalent (equal), if these functions are equal on a set of full measure.

We will solve the above formulated optimization problem (3) on the set  $\mathcal{KC}^1[0, T]$ , i.e.

$$J(u, t) \longrightarrow \inf_{u \in \mathcal{KC}^1[0, T], t \in [0, T]}. \quad (4)$$

The problem is that an optimal control does not exist always. For this reason a generalized control (lower or upper semicontinuous) is considered.

As an example, consider the following system of differential equations

$$\dot{x}(t) = u, \quad x(0) = 0,$$

and optimized functional is defined as

$$J(u) = \int_0^1 ((1 - u^2)^2 + x^2) d\tau \rightarrow \inf_{u \in \mathcal{KC}^1[0, 1]}.$$

This problem does not have an optimal control  $u(\cdot)$  in the set of piecewise continuously differentiable functions on  $[0, 1]$ , but has an optimizing sequence of controls  $\{u_k(\cdot)\}$  in the form of the piecewise continuous functions with values  $\pm 1$ . It is easy to see that the optimizing sequence  $\{x_k(\cdot)\}$ , corresponding to the sequence of controls  $\{u_k(\cdot)\}$ , has a limit  $x \equiv 0$  on  $[0, 1]$ . The control  $u(\cdot) \equiv 0$ , that corresponds to the solution  $x \equiv 0$ , can not be received as a pointwise limit of  $\{u_k(\cdot)\}$  and not an optimal control.

The right-hand side of the equation (1) can be very complex, and an exact solution of this equation can often be found approximately using the numerical methods. The optimization of the function  $J(\cdot)$  is also not easy if its form is complex. But optimization of the lower convex approximation of  $J(\cdot)$  is simpler. Moreover, the global optimum point does not disappear when we take the lower convex approximation. In addition, the construction of the lower convex approximation of  $J(\cdot)$ , that we define by  $\tilde{J}(\cdot)$ , makes it weakly lower semicontinuous, i.e.,

$$\lim_{u_k \rightarrow u} J(u_k) \geq J(u)$$

for any sequence  $\{u_k\}$  converging to  $u$  weakly, which is an important requirement for the weak convergence of an optimization sequence to a solution of the problem (2).

We propose here a method of an equivalent substitution with help of which we can come over these difficulties. Namely, it is suggested a substitution of the right-hand side of the equation (1) to another function with a simpler structure. The search for the solutions of the differential equation (1) (numerical or not) becomes simpler. The principle of an equivalent substitution declares that although we have another problem with simpler structure, but the function  $J(\cdot)$  attains the same infimum on the set of piecewise continuous differentiable functions. At the same time a new optimized functional  $\tilde{J}(\cdot)$  is getting lower semicontinuous.

Taking into consideration the said above about the substitution of the functions  $\varphi(\cdot)$  and  $J(\cdot)$ , we can conclude that the search for an optimal control and an optimal trajectory becomes easier and the new optimal problem is equivalent to the old one in the sense of finding an optimal control. In this case the necessary optimal condition is also sufficient.

## 2 The principle of equivalent replacement

Let us consider, as above, the system of differential equations (1), and optimized functional (4). We rewrite the system (1) and (4) in the form of differential equations

$$\begin{cases} \dot{x}(t) = \varphi(x(t), u(t)), \\ \dot{y}(t) = f(x(t), u(t)) \end{cases} \quad (5)$$

with the initial conditions  $x(0) = x_0$ ,  $y(0) = 0$ . The minimization problem (4) is replaced by the minimization problem

$$y(t, u) \longrightarrow \inf_{u \in \mathcal{KC}^1[0, T], t \in [0, T]}. \quad (6)$$

The solution of (5) is the solution of the integral equations

$$\begin{cases} x(t) = \int_0^t \varphi(x(\tau), u(\tau)) d\tau + x_0, \\ y(t) = \int_0^t f(x(\tau), u(\tau)) d\tau, \\ u(t) \in \mathcal{KC}^1[0, T], \quad t \in [0, T], \end{cases} \quad (7)$$

where  $u(\cdot)$  is a piecewise continuous differentiable control.

Let us unite all solutions of (7) in one set  $D(t)$  for  $t \in [0, T]$

$$\begin{aligned} D(t) = \{ (x, y, z) \mid x = x(t) = \int_0^t \varphi(x(\tau), u(\tau)) d\tau + x_0, \\ y = y(t) = \int_0^t f(x(\tau), u(\tau)) d\tau, z = u(t) \in \mathcal{KC}^1[0, T] \}, \end{aligned} \quad (8)$$

which is called the set of attainability of the systems (1) and (4) at time  $t \in [0, T]$ .

It is easy to see that the optimization problem (6) is equivalent to the following optimization problem

$$L(x, y, z) = y \longrightarrow \inf_{y \in \bigcup_{t \in [0, T]} D(t)}. \quad (9)$$

The function  $L(x, y, z) = y$  is linear in the coordinates  $(x, y, z)$ . (The function  $L(\cdot, \cdot, \cdot)$  depends only on the coordinate  $y$ ). It is known that any linear function reaches its maximum or minimum on the boundary of a compact set on which maximum or minimum is looking for.

Since the set of the solutions of (1) according to the assumptions is bounded on  $[0, T]$ , the set of the vector-functions  $u(\cdot)$  is closed and bounded in  $\mathcal{KC}^1[0, T]$ , then  $D(t)$  is closed and bounded for any  $t \in [0, T]$  in the metric  $\rho$  of the space  $\mathcal{KC}^1[0, T]$ . Indeed, if  $u_k \xrightarrow{\rho} u \in U$ , then, as it was mentioned above, there is the uniform convergence of  $u_k(\cdot)$  to  $u(\cdot)$  on  $[0, T] \setminus e$ , where  $e$  is a set with any small measure. Then there is convergence in measure [1]. The following convergence implies from the continuity of the functions  $\varphi(\cdot, \cdot)$ ,  $f(\cdot, \cdot)$  in all variables

$$\varphi(x(\tau), u_k(\tau)) \rightarrow_k \varphi(x(\tau), u(\tau)) \quad \forall \tau \in [0, T],$$

as well as

$$f(x(\tau), u_k(\tau)) \rightarrow_k f(x(\tau), u(\tau)) \quad \forall \tau \in [0, T].$$

The uniform convergence of the integrals with respect to  $t \in [0, T]$  and  $k$  implies from Egorov's theorem [2], i.e.

$$\int_0^t \varphi(x(\tau), u_k(\tau)) d\tau \rightarrow_k \int_0^t \varphi(x(\tau), u(\tau)) d\tau$$

and

$$\int_0^t f(x(\tau), u_k(\tau)) d\tau \rightarrow_k \int_0^t f(x(\tau), u(\tau)) d\tau.$$

Indeed, otherwise there exist sequences  $\{T_k\}$  and  $\{u_k(\cdot)\}$ , for which and for some  $\varepsilon > 0$ , the inequalities

$$\left| \int_0^{t_k} \varphi(x(\tau), u_k(\tau)) d\tau - \int_0^{t_k} \varphi(x(\tau), u(\tau)) d\tau \right| > \varepsilon$$

and

$$\left| \int_0^{t_k} f(x(\tau), u_k(\tau)) d\tau - \int_0^{t_k} f(x(\tau), u(\tau)) d\tau \right| > \varepsilon.$$

hold. The integrals can be considered as a function of  $t$ . It follows from Egorov's theorem that a set  $e$  exists with measure  $\mu(e) < \delta$  for any small  $\delta > 0$ , that the integrals, as the functions of  $t$ , will converge uniformly in  $k$  over the set  $[0, T] \setminus e$ . According to the absolute continuity, the integrals over the set  $e$  with measure  $\mu(e) < \delta$  will be arbitrarily small if  $\delta$  is also arbitrarily small. As a result, we get a contradiction with the existence of  $\varepsilon$ , for which the inequalities, written above, are true.

Thus we have proved

**Theorem 2.1** *The set  $D(t)$  is closed and bounded in  $KC^1[0, T]$  for any  $t \in [0, T]$  in the metric  $\rho$  of the space  $KC^1[0, T]$ .*

We form a sequence of functions defined on  $[0, T]$ ,

$$x_{k+1}(t) = \int_0^t \varphi(x_k(\tau), u_k(\tau)) d\tau. \quad (10)$$

The sequence  $\{x_k(\cdot)\}$  converges on  $[0, T]$  uniformly in  $k$ , if the sequence  $u_k(\cdot)$  converges in the metric  $\rho$  to the function  $u(\cdot)$  a.e. on  $[0, T]$ . Let us prove it.

Indeed, we know from the said earlier that the functions  $u_k(\cdot)$  converge to  $u(\cdot)$  uniformly on  $[0, T]$ . We replace the control  $u_k(\cdot)$  with the control  $u(\cdot)$  in (10). The difference between the original value of the integral (10) and the new value of the same integral can be evaluated in the following way. According to the inequality

$$|\varphi(x_k(\tau), u_k(\tau)) - \varphi(x_k(\tau), u(\tau))| \leq L |u_k(\tau) - u(\tau)| \quad \forall \tau \in [0, T],$$

where  $L$  is Lipschitz constant of the function  $\varphi(\cdot, \cdot)$  in the variables, the mentioned above difference is arbitrarily small for large  $k$  as well. Indeed, we have

$$\left| \int_0^t \varphi(x_k(\tau), u_k(\tau)) d\tau - \int_0^t \varphi(x_k(\tau), u(\tau)) d\tau \right| \leq L \int_0^t |u_k(\tau) - u(\tau)| d\tau \quad \forall t \in [0, T]$$

and the right side of this inequality is arbitrary small for large  $k$ .

We use the following result.

**Lemma 2.1** [3], [4]. *The sequence  $\{x_k(\cdot, u)\}, k = 1, 2, \dots$  converges uniformly for  $u \in U$  and  $k$  to the solution of (1).*

Summing up all said above, we can conclude about the uniform convergence on  $[0, T]$  of the solutions  $x_k(\cdot)$  of (1) for  $u = u_k(\cdot)$  to the solution  $x(\cdot)$  of the same system with the control  $u(\cdot)$  for  $k \rightarrow \infty$ .

**Lemma 2.2** *The sequence  $\{x_k(\cdot)\}, k = 1, 2, \dots$ , defined by (10), converges uniformly on  $[0, T]$  in  $k$  to the solution  $x(\cdot)$  of (1).*

**Remark 2.1** *Lemma (2.2) is also valid for the case when  $u_k \rightarrow u$  in the metric  $\rho_1$  of the space  $L_1[0, T]$ , i.e.*

$$\rho_1(u_k, u) = \int_0^T |u_k(\tau) - u(\tau)| d\tau.$$

The problem (9) has a solution if the functional (2) is lower semicontinuous. It will be shown how to do it.

If there is a solution of the problem (9) on the set piecewise continuous differentiable functions  $KC^1[0, T]$ , then we have solution of the problem

$$L(x, y, z) = y \longrightarrow \inf_{y \in \text{co } \bigcup_{t \in [0, T]} D(t)}, \quad (11)$$

where  $\text{co}$  is a symbol of taking the convex hull.

We introduce a set of attainability (or an attainability set) for the time  $T$ , which, by definition, is

$$D_T = \overline{\text{co}} \bigcup_{t \in [0, T]} D(t), \quad (12)$$

where  $\overline{\text{co}}$  means a closed convex hull. It is easy to see that for arbitrary

$$(x_k(t_k), y(t_k), u_k(t_k)) \in D(t_k)$$

such that

$$(x_k(t_k), y(t_k), u_k(t_k)) \rightarrow_k (x(t), y(t), u(t))$$

and

$$t_k \in [0, T], t \in [0, T], t_k \rightarrow_k t,$$

the inclusion

$$(x(t), y(t), u(t)) \in D(t)$$

will be true. Therefore, the closure in (12) can be removed and the definition of the set  $D_T$  can be given as following

$$D_T = \text{co } \bigcup_{t \in [0, T]} D(t).$$

Moreover, the problem (9) and (11) are equivalent in the sense that if one of them has a solution, then the other one has a solution as well and these solutions are the same. In addition, since projections of the set  $D(t), t \in [0, T]$  on the axis  $x, y$  are closed and bounded, and, hence, compact in the corresponding finite-dimensional spaces and  $D(\cdot)$  is continuous in  $t$  as the set-valued mapping, then  $\inf$  in (11) can be replaced by  $\min$  and the problem (11) can be rewritten in the following way

$$L(x, y, z) = y \longrightarrow \min_{y \in D_T} . \quad (13)$$

But the problem (13) would be the same if we change the functions  $\phi(\cdot, \cdot)$  and  $f(\cdot)$  on their lower convex approximations constructed on the set of attainability for the time  $T$ . Indeed, we take two arbitrary points  $(x_1(t), y_1(t), u_1)$  and  $(x_2(t), y_2(t), u_2)$ ,  $t \in [0, T]$  from set  $D(t)$ . Consider a linear combination with the nonnegative coefficients  $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 = 1$ . Then, the point  $(\alpha_1 x_1(t) + \alpha_2 x_2(t), \alpha_1 y_1(t) + \alpha_2 y_2(t), \alpha_1 u_1 + \alpha_2 u_2)$  will belong to the set  $\text{co } D(t)$ , if we replace the functions  $\varphi(\cdot, \cdot)$  and  $f(\cdot, \cdot)$  with the following:

$$\tilde{\varphi}(\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau), \alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \varphi(x_1(\tau), u_1) + \alpha_2 \varphi(x_2(\tau), u_2)$$

and

$$\tilde{f}(\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau), \alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 f(x_1(\tau), u_1) + \alpha_2 f(x_2(\tau), u_2).$$

But this construction, built for all points of the regions  $D(t), t \in [0, T]$ , just means that we are constructing the lower convex approximations of the functions  $\varphi(\cdot, \cdot)$  and  $f(\cdot, \cdot)$  in the attainability set for the time  $T$ , i.e. in  $D_T$ .

Denoted by  $\tilde{J}(\cdot, \cdot)$  the new optimization function after replacement of the function  $f(\cdot, \cdot)$  with  $\tilde{f}(\cdot, \cdot)$  in  $D_T$

$$\tilde{J}(u, t) = \int_0^t \tilde{f}(x(\tau), u(\tau)) d\tau.$$

It is clear that  $\tilde{J}(\cdot, \cdot)$  takes the same optimal value in the attainability set  $D_T$ , that the functional (4)  $J(\cdot, \cdot)$  takes for the system (1).

Replace the system (1) on the system

$$\dot{x}(t) = -\varphi(x(t), u(t)), \quad x(0) = -x_0, \quad (14)$$

and the optimized functional with the functional

$$J(u, t) = \int_0^t f(-x(\tau), u(\tau)) d\tau. \quad (15)$$

It is easy to see that the minimum or the maximum of the functional  $J(\cdot, \cdot)$  did not change. Hence, the problem (1), (4) and (14), (15) are replaceable. So "convexification" of the function  $\varphi(\cdot, \cdot)$ , in contrast to the procedure of "convexification" of the function  $f(\cdot, \cdot)$  should be as following:

1. We construct the lower convex approximation of the function  $\varphi(\cdot, \cdot)$  in the variables  $(x, u)$  from the attainability set for the time  $T$ , i.e.  $D_T$ , which we denote by  $\tilde{\varphi}_1(\cdot, \cdot)$ ;
2. Construct the upper concave approximation of the function  $\varphi(\cdot, \cdot)$  (or, equivalently, we construct the lower convex approximation for the function  $-\varphi(\cdot, \cdot)$  and after that take minus of this function) in the variables  $(x, u)$  from the attainability set for the time  $T$ , i.e.  $D_T$ , which we denote by  $\tilde{\varphi}_2(\cdot, \cdot)$ ;
3. Let us replaced the system (1), (4) by two systems of the equations:

$$\dot{x}(t) = \tilde{\varphi}_1(x(t), u(t)), \quad x(0) = x_0, \quad u(\cdot) \in \mathcal{KC}^1[0, T] \quad (16)$$

with the optimization function  $\tilde{J}(u, t)$  and

$$\dot{x}(t) = \tilde{\varphi}_2(x(t), u(t)), \quad x(0) = x_0, \quad u(\cdot) \in \mathcal{KC}^1[0, T] \quad (17)$$

with the same optimization function  $\tilde{J}(u, t)$ ;

4. Let us find among the solutions of (16) and (17) the one that gives the smallest value of the functional  $\tilde{J}(u, t)$  in  $D_T$ .

We obtain the following result.

**Theorem 2.2** *There are the solutions among the solutions of (16) and (17) such that deliver a minimum (maximum) in  $u(\cdot) \in \mathcal{KC}^1[0, T]$  and  $t \in [0, T]$  for the functional*

$$\tilde{J}(u, t) = \int_0^t \tilde{f}(x(\tau), u(\tau)) d\tau,$$

*that coincides with the infimum (supremum) of the functional  $J(u, t)$  (see (4)). In this case the necessary conditions for a minimum (maximum) are also sufficient conditions.*

**Remark 2.2** *The set  $D(t), t \in [0, T]$  is not necessarily compact, although its projection on the axis  $x, y$  are compact. That's why we were able to move to the problem*

$$L(x, y, z) = y \longrightarrow \min_{y \in D_T},$$

*if the problem (9) has a solution. The latter coincides with the formulation of Mazur's theorem. It asserts that in any weakly convergent sequence  $\{u_k(\cdot)\} \in L_p([0, T])$ ,  $u_k(\cdot) \longrightarrow u(\cdot)$  can be chosen for each  $k$  subsequence convex hull of which is almost everywhere on  $[0, T]$  for  $k \rightarrow \infty$  converges to some  $u(\cdot) \in L_p[0, T]$ . In our case, there exists a sequence  $\{u_k(\cdot)\} \in \mathcal{KC}^1[0, T]$ , the convex hull of which will converge to an optimal control  $u(\cdot) \in \mathcal{KC}^1[0, T]$ . The sequence of the solutions  $\{x_k(\cdot)\}$ , corresponding to the controls  $u_k(\cdot)$ , will converge to an optimal solution  $x(\cdot)$ , corresponding to the control  $u(\cdot)$ , provided that the solutions have been calculated for the problems with the modified right-hand side.*



Consider a few examples. It is clear that an equivalent replacement of one system to another can be applied to the differential system without the control  $u$ .

Example 1. Let us given a differential equation

$$\dot{x}(t) = \varphi(x(t)) = \begin{cases} (X - 1)^2, & \text{if } x \geq 0 \\ (X + 1)^2, & \text{if } x < 0 \end{cases}$$

with initial condition  $x(0) = 0$ . The optimized functional is given by

$$f(x(t)) = x^2(t) \rightarrow \min \quad \text{for } t \in (-\infty, +\infty),$$

The general solution of differential equations for  $x \geq 0$  has the form

$$x(t) = -\frac{1}{t+c} + 1,$$

which tends to 1 for  $t \rightarrow \infty$ . The general solution of the differential equation for  $x < 0$  is given by

$$x(t) = -\frac{1}{t-c} - 1,$$

which tends to  $-1$  for  $t \rightarrow \infty$ . In order to meet the initial condition we have to put  $c = 1$ . The projection of the attainability set  $D_{(-\infty, +\infty)}$  on  $OX$  axis is the interval  $(-1, +1)$ .

It is clear that the function  $f(\cdot)$  takes its minimum at  $x = 0$ . But the same solution is obtained if instead of the function  $\varphi(\cdot)$  to take its lower convex approximation, namely, the function

$$\tilde{\varphi}(x) = \begin{cases} (X - 1)^2, & \text{if } x \geq 1 \\ 0, & \text{if } -1 \leq x \leq 1 \\ (X + 1)^2, & \text{if } x < -1. \end{cases}$$

Example 2. The same example, but

$$J(x, u) = \int_0^t x^2(\tau) d\tau,$$

which we minimize for  $t \in [0, 1]$ . The solution has the form

$$x(t) = -\frac{1}{t+1} + 1.$$

Here the replacement of the function  $\varphi(\cdot)$  with the function  $\tilde{\varphi}(\cdot)$  on the whole line is impossible, since the projection of the attainability set  $D_1$  on  $OX$  axis is the interval  $[0, 1/2]$ .

Example 3. Let us given the differential equation

$$\dot{x}(t) = x^2$$

with the initial condition  $x(0) = 1$ . The general solution has the form

$$x(t) = -\frac{1}{t+c},$$

a solution, satisfying the initial condition, is

$$x(t) = -\frac{1}{t-1}.$$

The optimized functional has the form

$$J(x, u) = \int_0^t (-x^2(\tau))d\tau \longrightarrow_t \inf$$

for  $t \in [0, 1]$ .

It is easy to compute its optimal value

$$J(x, u) = \int_0^1 (-x^2(\tau))d\tau = \int_0^1 (-\dot{x}(\tau))d\tau = x(0) - x(1) = -\infty.$$

In this case, the projection of the attainability set  $D_1$  on  $OX$  axis is the set  $(-\infty, 0) \cup [1, +\infty)$ . It is easy to see that the lower convex approximation of the functional  $J(\cdot)$  on  $D_1$ , which we denote by  $\tilde{J}(\cdot)$ , takes the same value. Similar is true for the functional

$$J(x, u) = \int_1^\infty (-x^2(\tau))d\tau = \int_1^\infty (-\dot{x}(\tau))d\tau = x(1) - x(+\infty) = -\infty.$$

Example 4. Let us consider the following problem

$$\dot{x}(t) = x \sin(1/x) + u, \quad x(0) = 0.$$

The optimized functional is

$$J(u) = \int_0^\infty |u(\tau) - x(\tau)| d\tau \rightarrow \inf_u.$$

We will get the following system after construction of the lower and upper approximations

$$\dot{x}(t) = x + u, \quad x(0) = 0$$

and

$$\dot{x}(t) = -x + u, \quad x(0) = 0.$$

The optimal solution exists among their solutions  $x(t) \equiv u(t) = 0$ .

### 3 An evaluation of the attainability set

Let us have a system of differential equations

$$\dot{x}(t) = \varphi(x, u), \quad x \in \mathbb{R}^n, \quad t \in [0, T], \quad u(t) \in U \subset \mathbb{R}^r \quad (18)$$

with the initial condition  $x(0) = 0$ , where  $\varphi(\cdot, \cdot)$  is Lipschitz in the variables  $x, u$ ,  $U$  is a convex compact set in  $\mathbb{R}^r$ . The problem is to estimate the attainability set. By definition, the area of attainability for the time  $T$  is the set

$$D_T = \overline{c0} \cup_{t \in [0, T]} D(t),$$

where

$$D(t) = \{x \in \mathbb{R}^n \mid x = x(t) = \int_0^t \varphi(x(\tau), u(\tau)) d\tau, \quad u(\tau) \in U, u(\cdot) \in \mathcal{KC}^1[0, T]\}.$$

The choice of the initial position and the initial time of zero is not loss of generality.

Take an arbitrary positively definite function  $V(x)$  (see [5]), satisfying the condition

$$m_1 \|x\|^2 \leq V(x) \leq m_2 \|x\|^2.$$

Let be

$$\varphi(x, u, t) = \varphi_1(x, u, t) + \varphi_2(x, u, t)$$

and  $v(\cdot) : [0, T] \rightarrow \mathbb{R}^n$  is a piecewise continuous vector-function.

Consider the systems of differential equations

$$\dot{x}(t) = \tilde{\varphi}_1(x, u) = \varphi_1(x, u) + v(t), \quad t \in [0, T], \quad (19)$$

and

$$\dot{x}(t) = \tilde{\varphi}_2(x, u) = \varphi_2(x, u) - v(t), \quad t \in [0, T], \quad (20)$$

Denote by

$$D_T^{(i)} = \overline{c0} \cup_{t \in [0, T]} D_i(t), \quad i = 1, 2, \quad (21)$$

the attainability sets for the systems (19), (20), where

$$D_i(t) = \{x \in \mathbb{R}^n \mid x = x_i(t) = \int_0^t \tilde{\varphi}_i(x(\tau), u(\tau)) d\tau, \quad u(\tau) \in U, \\ u(\cdot) \in \mathcal{KC}^1[0, T]\}, \quad i = 1, 2. \quad (22)$$

Let the estimates of the attainability sets for the time  $T$  be given respectively by the inequalities

$$0 \leq V(x) \leq c_1, \quad 0 \leq V(x) \leq c_2.$$

We get an estimation for the attainability set of the system (18) for the time  $T$ . We show that the attainability set  $D_T$  for this system satisfies the inclusion

$$D_T \subset D_T^{(1)} + D_T^{(2)}.$$

Indeed, by definition, the set  $D_T^{(1)} + D_T^{(2)}$  will consist of the points on the curves tangent to which is the sum of the tangents to the curves consisting of the points of the sets  $D_1(t_1)$  and  $D_2(t_2)$  for all  $t_1, t_2 \in [0, T]$ . It is clear, that for some vector-function  $v(\cdot)$  the resulting set will include  $D_T$ , which consists of the points on the curves tangent to which is the sum of the tangents to the curves consisting of the points of the sets  $D_1(t)$  and  $D_2(t)$  for all  $t \in [0, T]$ .

As a result, the following theorem is proved.

**Theorem 3.1** *For the attainability set  $D_T$  of (18) the inclusion*

$$D_T \subset D_T^{(1)} + D_T^{(2)},$$

*is true, where  $D_T^{(i)}$ ,  $i = 1, 2$ , are given by (21), (22) .*

It follows from here the following lemma.

**Lemma 3.1** *The function  $V(\cdot)$  satisfies the inequality*

$$0 \leq V(x) \leq c_1 + c_2$$

*in the attainability set  $D_T$  of the system (18).*

Now consider two systems of differential equations (18) and (??) with zero initial conditions at time zero. Let it be known that, for all  $x, u$  the inequality

$$0 < k_1 \leq \left\| \frac{\varphi(x, u)}{\varphi_1(x, u)} \right\| \leq k_2 \quad (23)$$

is true. Let us also know the attainability set  $D_T^{(1)}$  of the system (??) for the time  $T$ . The problem is to obtain some estimates of the attainability set of the system (18).

The arguments will be carried out as before, considering the trajectories of the systems (18) and (??).

Any vector in the set  $D_T$  for some  $t \in [0, T]$  and  $u \in \mathcal{KC}^1[0, T]$  is

$$x = x(t) = \int_0^t \varphi(x(\tau), u(\tau)) d\tau.$$

Consequently,

$$\|X(t)\| = \left\| \int_0^t \varphi(x(\tau), u(\tau)) d\tau \right\| \leq \int_0^t \|\varphi(x(\tau), u(\tau))\| d\tau \leq$$

$$\leq k_2 \int_0^t \|\varphi_1(x(\tau), u(\tau))\| d\tau \subset k_2 D_1(t).$$

Since the previous inclusion holds for any  $t \in [0, T]$ , it follows that

$$D_T \subset \cup_{t=0}^T k_2 D_1(t) = k_2 \cup_{t=0}^T D_1(t) \subset k_2 \overline{CO} \cup_{t=0}^T D_1(t) = k_2 D_T^{(1)}.$$

The theorem is proved.

**Theorem 3.2** *For the systems (18) and (??) with the attainability sets  $D_T$  and  $D_T^{(1)}$  respectively, for which the inequality (23) holds, the inclusion*

$$D_T \subset k_2 D_T^{(1)}$$

*is true.*

From here we can easily obtain the following conclusion.

**Lemma 3.2** *In the attainability set  $D_T$  of the system (18) the function  $V(\cdot)$  satisfies the inequality*

$$0 \leq V(x) \leq k_2 c_1,$$

*where the constant  $c_1$  limits the top value of the function  $V(\cdot)$  in the attainability set  $D_T^{(1)}$  of the system (??).*

## 4 Conclusion

The obtained results allow us to move from a local optimal problem to a global optimal problem. To implement this it is required to construct the lower convex approximation and the upper concave approximation of the function written on the right side of the system of differential equations. We also construct the lower convex approximation for the optimized function. All constructions are made in the attainability set for the time  $T$ .

It is suggested a method for an estimation of the attainability set with help of the positively definite functions (Lyapunov functions). The proposed method is based on the decomposition of the function, stayed on the right side of the system of differential equations, into the components for which the attainability set or its estimates are already known. That makes difference from the paper [6], where the linear systems are considered.

The proposed transformation method of the systems is particularly useful when it is difficult to get a solution of differential equations in an explicit form, but only using approximate methods. In addition, the sufficient conditions of optimality holds for an optimal control obtained according to the proposed method.

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